Fibonacci sequence, golden section, Kalman filter and optimal control

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Abstract

A connection between the Kalman filter and the Fibonacci sequence is developed. More precisely it is shown that, for a scalar random walk system in which the two noise sources (process and measurement noise) have equal variance, the Kalman filter's estimate turns out to be a convex linear combination of the *a priori* estimate and of the measurements with coefficients suitably related to the Fibonacci numbers. It is also shown how, in this case, the steady-state Kalman gain as well as the predicted and filtered covariances are related to the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$. Furthermore, it is shown that, for a generic scalar system, there exist values of its key parameters (i.e. system dynamics and ratio of process-to-measurement noise variances) for which the previous connection is preserved. Finally, by exploiting the duality principle between control and estimation, similar connections with the linear quadratic control problem are outlined.

Keywords: *Kalman filter; Linear Quadratic Control; golden section/ratio; Fibonacci sequence; random walk.*

1 Introduction

The Fibonacci numbers and the golden section are ubiquitous in nature and art as well as in science and engineering. Is there any link between the Nautilus shell, the layout of the sunflower's seeds and the proportions of the Botticelli's Venus ? The answer to this question is the irrational number $\phi = 1.618033988749895...$, known as *golden ratio* [1], which is perhaps the first irrational number ever discovered. The number ϕ has interesting properties. Its square is equal to $\phi+1$, its reciprocal (called *golden section*) is equal to $\phi-1$ and, further, ϕ is intimately connected to the sequence of Fibonacci numbers $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$. The Fibonacci sequence, starting from 0 and 1, is defined by recurrence by taking each subsequent number as the sum of the two previous ones. It can easily be checked that the ratio between a number of the sequence and the previous one converges to the golden ratio.

The goal of this paper is to show the existence of a connection between the Fibonacci sequence and the golden section/ratio on one side and the Kalman filter on the other side. By applying the duality principle between estimation and control, a connection with the linear quadratic control problem is also derived. Before proceeding, it should be pointed out that this connection must be regarded as a mere mathematical curiosity which, however, makes the history of ϕ even more fascinating.

The rest of the paper is organized as follows. Section 2 reviews some historical notes; section 3 displays the connection between the Fibonacci sequence and the Kalman filter and, finally, section 4 lists the conclusions.

2 Historical notes

2.1 Golden Section

The first mathematical definition of *Golden Ratio* traces back to the famous Greek mathematician Euclid who, in the 3d century B.C., introduced it [2] to solve a geometrical problem called the problem of *division of a line segment in extreme and mean ratio*. The essence of the problem is the following. A line segment AB must be divided with a point C into two parts so that the ratio between the longer part CB and the shorter one AC is equal to the ratio between the whole line segment AB and the longer part CB, i.e.:

$$\frac{AB}{CB} = \frac{CB}{AC} \tag{2.1}$$

Exploiting the relationship $AB = AC + CB$, equation (2.1) can be written in the following form:

$$x = \frac{CB}{AC} = \frac{AB}{CB} = \frac{AC + CB}{CB} = 1 + \frac{AC}{CB} = 1 + \frac{1}{x} \tag{2.2}$$

Hence, the equation to calculate the ratio x is:

$$x^2 - x - 1 \ = \ 0 \tag{2.3}$$

The positive root of the equation (2.3) is the solution of the problem of *division of a line segment in extreme and mean ratio*. This solution is just the *golden ratio*:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \tag{2.4}$$

The discovery of the number ϕ, however, cannot be attributed to Euclid but probably to Pythagoras or to Hippasus of Metapontum (a disciple of Pythagoras) in the 5th century B.C. [1]. In fact, the ancient philosopher Pythagoras chose the pentagram (a regular star with 5 vertices) as the symbol of the secret fraternity of which he was both leader and founder and the construction of a pentagram is based on the golden ratio. The pentagram can be seen as a geometric shape consisting of 5 straight lines arranged as a star with 5 points. The intersection of the lines naturally divides each line into 3 parts. The smaller part (which forms the pentagon inside the star) is proportional to the longer parts (which form the points of the star) by a ratio of $1 : \phi$. For the connection between Pythagoreans and the pentagram it is reasonable to believe that the Pythagoreans were the first to discover ϕ and so the first to discover the existence of irrational numbers. The mathematical history of ϕ went on and today many interesting things about this number are known. A list of the mathematical properties of ϕ is given below:

- ϕ can be expressed as a continuous fraction with only 1;

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}$$

- ϕ can be expressed as a continuous square root;

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$$

- Any power of ϕ is equal to the sum of the two immediately preceding powers:

$$\phi^n = \phi^{n-1} + \phi^{n-2}.$$

It is worth pointing out that ϕ has many other mathematical and geometrical properties, but the most interesting is certainly the connection with the Fibonacci sequence.

2.2 Leonardo Pisano Fibonacci

Leonardo Pisano (1170-1250), better known by his nickname Fibonacci, was born in Italy but was educated in North Africa where his father, Guilielmo, held a diplomatic post [3]. His father's job was to represent the merchants of the Republic of Pisa who were trading in Bugia, later called Bougie and now called Bejaia (Algeria). Fibonacci was taught mathematics in

Figure 1: Leonardo Pisano Fibonacci

Bugia and travelled widely with his father and recognised the enormous advantages of the mathematical systems used in the countries they visited.

Fibonacci ended his travels around the year 1200 and at that time he returned to Pisa [3]. There he wrote a number of important texts which played an important role in reviving ancient mathematical skills and he made significant contributions of his own.

The most important book of Fibonacci, Liber abaci (the book of calculations) [4], was published in 1202 after Fibonacci's return to Italy. The book was based on the arithmetic and algebra that Fibonacci had accumulated during his travels. The book, which went on to be widely copied and imitated, introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe [3].

The book begins with the following premise: "*The nine Indian figures*[1] *are:* $9, 8, 7, 6, 5, 4, 3, 2, 1$. *With these figures, and with the sign 0... any number may be written, as is demonstrated below.*"

Although *Liber Abaci* is mainly a book about the use of Arab numerals, it contains also

[1] Arabic numerals

the solution of important problems. In fact, the second section of Liber abaci contains a large collection of problems aimed at merchants. They relate to the price of goods, how to calculate profit on transactions, how to convert between the various currencies in use in Mediterranean countries, and problems which had originated in China [3]. A problem in the third section of Liber abaci led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today: "*A certain man puts a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?* " The resulting sequence, in which each number is the sum of the two preceding numbers, is $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

The Fibonacci sequence was forgotten until the famous astronomer Kepler rediscovered the sequence in the 1611 A.D. Further, he made a remarkable discovery about the ratios of consecutive terms of the Fibonacci sequence. He guessed that these ratios approximate what he calls the *divine proportion* and what is now called the *golden ratio*. That is he saw that the ratios 1/1, 2/1, 3/2, 5/3, ... rapidly approach $\phi = 1.618\ldots$.

The Fibonacci sequence has proved extremely fruitful and appears in many different areas of mathematics and science. Some interesting mathematical properties are given below.

- The sequence of final digits in Fibonacci numbers repeats in cycles of 60.
- The last two digits repeat in cycles of 300, the last three in 1500, the last four in $15,000$, etc.
- Let f_i be the i-th element of the Fibonacci sequence then:

  1. f_n divides f_{nm} for any positive integers n and m;

Figure 2: Sunflower

2. $gcd(f_n, f_m) = f_{gcd(m,n)}$, where gcd is the greatest common divisor;

3. $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$

Fibonacci numbers seem to appear also in nature. For example, many types of flowers have a Fibonacci number of petals: daisies tend to have 34 or 55 petals, sunflowers have 89 or 144 [12]. Similarly, the numbers of rings on the trunks of palm trees, the scales on the surface of a pineapple and the genealogy of the male bee all follow a sequence of Fibonacci numbers. The arrangement of plant leaves, or phyllotaxis, unfolds to the same pattern because this seems to be an optimal solution in terms of the spacing of the leaves or the amount of light that can reach them.

The history of the Fibonacci numbers and the golden ratio is also related to art and architecture. The golden section seems to appear in many of the proportions of famous ancient buildings, as the Parthenon in Athens or Palazzo della Signoria in Florence [13], Also the proportions of famous paintings seem to be designed according to the golden section, for examples the Botticelli's Venus in the painting *La Primavera* or the *Vergine delle Rocce* of Leonardo Da Vinci. However, there is no original documentary evidence that these buildings and paintings were deliberately designed using the golden section.

3 Fibonacci sequence, Kalman filter and optimal control

A Fibonacci characterization of the *best linear unbiased* estimator has been presented in [5] by following a non recursive least-squares approach. Hereafter a more complete characterization is provided by following a recursive Riccati equation approach, which is the fundamental core of the Kalman algorithm and Linear Quadratic (LQ) control.

3.1 Kalman Filter and Fibonacci sequence

Consider the scalar stochastic dynamical system

$$\begin{cases} x(t+1) = x(t) + w(t) \\ y(t) = x(t) + v(t) \end{cases} \tag{3.1}$$

with process noise $w(t)$ and measurement noise $v(t)$ having the same variance σ^2. Let us hereafter denote by $p(t|t-1)$ and $p(t|t)$ the *a priori* and, respectively, *a posteriori* state variance normalized by the common noise variance σ^2.

For the system (1), the Kalman filter propagates the estimate according to [6]

$$\hat{x}(t) = \hat{x}(t-1) + k(t)\left[y(t) - \hat{x}(t-1)\right] = \left[1 - k(t)\right]\hat{x}(t-1) + k(t)y(t) \tag{3.2}$$

where the Kalman gain is given by

$$k(t) = \frac{p(t|t-1)}{1+p(t|t-1)}. \tag{3.3}$$

Further, the state covariance is updated by:

$$\begin{aligned} p(t|t) &= p(t|t-1) - \frac{p^2(t|t-1)}{1+p(t|t-1)} = \frac{p(t|t-1)}{1+p(t|t-1)} = k(t) \\ p(t+1|t) &= 1 + p(t|t) = 1 + \frac{p(t|t-1)}{1+p(t|t-1)} \end{aligned} \tag{3.4}$$

Let us now introduce the Fibonacci sequence:

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_k = f_{k-1} + f_{k-2} \quad \forall k \geq 2 \end{cases} \tag{3.5}$$

It is easy to see that, given the prior state variance $p(1|0) = p_0$, from (3.3) and (3.4) one gets

$$k(t) = p(t|t) = \frac{f_{2t-2} + f_{2t-1}\ p_0}{f_{2t-1} + f_{2t}\ p_0} \qquad \forall t \geq 1 \tag{3.6}$$

In fact,

$$k(1) = p(1|1) = \frac{p_0}{1+p_0} = \frac{f_0 + f_1\ p_0}{f_1 + f_2\ p_0}$$

Next, by induction, if (3.6) holds then

$$\begin{aligned} k(t+1) &= \frac{p(t+1|t)}{1+p(t+1|t)} = \frac{1+p(t|t)}{2+p(t|t)} = \frac{1+k(t)}{2+k(t)} \\ &= \frac{f_{2t-1}+f_{2t-2}+(f_{2t-1}+f_{2t})\,p_0}{2f_{2t-1}+f_{2t-2}+(2f_{2t}+f_{2t-1})\,p_0} = \frac{f_{2t}+f_{2t+1}p_0}{f_{2t+1}+f_{2t+2}p_0} \end{aligned}$$

i.e. (3.6) holds also for $t+1$ and hence for all $t \geq 1$. It is well known that the Fibonacci sequence satisfies the limit properties:

$$\begin{aligned} \lim_{i\to\infty} \frac{f_{i-1}}{f_i} &= \alpha = \frac{\sqrt{5}-1}{2} \approx 0.618 \\ \lim_{i\to\infty} \frac{f_{i+1}}{f_i} &= \phi = \frac{1}{\alpha} = 1+\alpha = \frac{\sqrt{5}+1}{2} \approx 1.618 \end{aligned} \tag{3.7}$$

where α and ϕ are the so called *golden section* and, respectively, *golden ratio*. Using the properties (3.7), it can be checked that, irrespective of the initial condition p_0,

$$\begin{aligned} k_\infty &= \lim_{t\to\infty} k(t) = \lim_{t\to\infty} p(t|t) = \alpha \\ p_\infty &= \lim_{t\to\infty} p(t+1|t) = 1+\alpha = \phi \end{aligned} \tag{3.8}$$

i.e. the Kalman gain and the normalized *a posteriori* state variance converge to the golden section α while the normalized *a priori* state variance converges to the golden ratio ϕ. The same result could be obtained by finding the positive solution of the *algebraic Riccati equation*

$$p = 1+\frac{p}{1+p} \implies p^2-p-1=0 \implies p_\infty = \frac{\sqrt{5}+1}{2} \stackrel{\triangle}{=} \phi \implies k_\infty = \frac{\sqrt{5}-1}{2} \stackrel{\triangle}{=} \alpha$$

It can be checked that, propagating (3.2) forward in time, one gets the following expression for the estimate $\hat{x}(t)$ in terms of the initial estimate $\hat{x}(0)$ and the measurements $\{y(1), y(2), \ldots, y(t)\}$:

$$\hat{x}(t) \;=\; w_0(t)\hat{x}(0) + \sum_{i=1}^{t} w_i(t)y(i) \tag{3.9}$$

where the weights $w_i(t)$ of the data fusion at time t are given by

$$\begin{aligned} w_0(t) &= \frac{1}{f_{2t-1} + f_{2t}p_0} \\ w_i(t) &= \frac{f_{2i-2} + f_{2i-1}p_0}{f_{2t-1} + f_{2t}p_0} \qquad i = 1, 2, \ldots, t \end{aligned} \tag{3.10}$$

In fact, (3.10) can be proved by induction. It holds true for $t = 1$ since

$$\hat{x}(1) \;=\; [1 - k(1)]\,\hat{x}(0) + k(1)y(1) \;=\; \frac{1}{f_1 + f_2p_0}\hat{x}(0) + \frac{f_0 + f_1p_0}{f_1 + f_2p_0}y(1)$$

Further, if (3.9) holds then

$$\begin{aligned} \hat{x}(t+1) &= [1 - k(t+1)]\,\hat{x}(t) + k(t+1)y(t+1) \\ &= [1 - k(t+1)]\,w_0(t)\hat{x}(0) + \sum_{i=1}^{t} [1 - k(t+1)]\,w_i(t)y(i) \;+\; k(t+1)y(t+1) \\ &= w_0(t+1)\hat{x}(0) + \sum_{i=1}^{t+1} w_i(t+1)y(i) \end{aligned}$$

since

$$\begin{aligned} w_i(t+1) &= [1 - k(t+1)]\,w_i(t) \qquad i = 0, 1, \ldots, t \\ w_{t+1}(t+1) &= k(t+1) \end{aligned}$$

In the case in which no *a priori* information on the initial state is available, i.e. $p_0 \to \infty$, it turns out from (3.10) that $w_0(t) \to 0$ and $w_i(t) \to f_{2i-1}/f_{2t}$; hence

$$\text{for } p_0 \to \infty : \quad \hat{x}(t) \to \sum_{i=1}^{t} \frac{f_{2i-1}}{f_{2t}}\, y(i) \tag{3.11}$$

wherein the coefficients of the data fusion are ratios of Fibonacci numbers.

Summarizing the above connection between Kalman filtering, Fibonacci sequence and golden section/ratio, related to the system (3.1) with process noise covariance equal to the measurement noise covariance, the following facts hold.

1. The steady-state Kalman filter turns out to be

$$\hat{x}(t) \;=\; (1-\alpha)\,\hat{x}(t-1) + \alpha y(t)$$

   where $\alpha \triangleq \frac{1}{2}\left(\sqrt{5}-1\right) \approx 0.618$ is the golden section, and the associated steady-state filtered covariance is α times the noise covariance.

2. The steady-state predicted covariance converges to ϕ times the noise covariance, where $\phi = \alpha^{-1} \triangleq \frac{1}{2}\left(\sqrt{5}+1\right) \approx 1.618$ is the golden ratio.

3. In general, the current estimate $\hat{x}(t)$ is related to the initial estimate $\hat{x}(0)$, to the measurements $\{y(1), y(2), \ldots, y(t)\}$ and to the initial covariance p_0 by the expression

$$\hat{x}(t) \;=\; \frac{1}{f_{2t-1} + f_{2t}p_0}\,\hat{x}(0) \;+\; \sum_{i=1}^{t} \frac{f_{2i-2} + f_{2i-1}p_0}{f_{2t-1} + f_{2t}p_0}\, y(i)$$

   which involves the Fibonacci sequence $\{f_k\}$.

4. Provided that the initial covariance is large, i.e. $p_0 \to \infty$, the current estimate $\hat{x}(t)$

is a combination of all measurements $y(i)$ with coefficients f_{2i-1}/f_{2t} equal to ratios of Fibonacci numbers.

The previous results can be extended to a generic first order system:

$$\begin{cases} x(t+1) & = & a\, x(t) \; + \; w(t) \\ y(t) & = & x(t) \; + \; v(t) \end{cases} \tag{3.12}$$

where the process noise $w(t)$ and measurement noise $v(t)$ have generic variances q and, respectively, r. The objective is to investigate for which values of the parameters a, q, r the steady-state Kalman filter correction (measurement update)

$$\hat{x}(t|t) \;=\; (1-\ell)\;\hat{x}(t|t-1) \;+\; \ell\, y(t)$$

has gain ℓ equal to either the golden section α or its complement $1-\alpha$. This amounts to weighting the predicted estimate and the current measurement according to a *golden section*-like data fusion. Using the gain formula $\ell = p(r+p)^{-1}$ and the Riccati equation

$$p^2 + \big(r - ra^2 - q\big)\, p - qr \;=\; 0$$

it is a straightforward exercise to show what follows.

1. For the variance ratio $r/q > \phi$, or equivalently $q/r < \alpha$, there exist two symmetric values

$$a \;=\; \pm\, \sqrt{(1+\alpha)\left(1 - \frac{1}{\alpha}\,\frac{q}{r}\right)} \tag{3.13}$$

such that the resulting correction gain is $\ell = 1 - \alpha$ and, hence, the estimate correction is

$$\hat{x}(t|t) \;=\; \alpha\; \hat{x}(t|t-1) \;+\; (1-\alpha)\; y(t).$$

2. For the variance ratio $r/q > \alpha$, or equivalently $q/r < \phi$, there exist two symmetric values

$$a \;=\; \pm\; \sqrt{(1+\phi)\left(1 - \frac{1}{\phi}\,\frac{q}{r}\right)} \tag{3.14}$$

such that the resulting correction gain is $\ell = \alpha$ and, hence, the estimate correction is

$$\hat{x}(t|t) \;=\; (1-\alpha)\; \hat{x}(t|t-1) \;+\; \alpha\; y(t).$$

Notice that the case so far considered of equal variances, i.e. $r/q = 1 > \alpha$, corresponds to the latter situation. From (3.14) it turns out that the golden section update is obtained only for the two values $a = \pm 1$, of which $a = 1$ corresponds to the studied random walk system. Further notice that for $r/q < \alpha$, i.e. for a sufficiently small variance of the measurement noise compared to the variance of the process noise, a golden section data fusion is not possible for any value of the parameter a.

3.2 Linear Quadratic Control

It is well known that the Linear Quadratic (LQ) control and the Kalman filter are dual mathematical problems. This result was first described in [6]. Therefore, obviously, the previous results hold also for the LQ control problem when the dual systems of (3.1) or (3.12) are considered. For instance, the dual of the estimation problem given in section 3.1

can be formulated as follows. Given the dynamical system

$$x(t+1) \quad = \quad x(t) + u(t) \tag{3.15}$$

find the admissible control strategy $\{u(t) : \; 0 \leq t < N\}$, which drives the initial state $x(0)$ to the final state $x(N)$, minimizing the following quadratic loss function:

$$J_N = q_0 \, x^2(N) + \sum_{t=0}^{N-1} \gamma^2 (x^2(t) + u^2(t)) \tag{3.16}$$

Notice that, in the dual problem, the parameters $q_0 > 0$ and γ^2 play the same role as p_0 and, respectively, σ^2 defined in section 3.1. An admissible control strategy is such that $u(t)$ is a function of the information available at time instant t only. Assuming that the state $x(t)$ is fully known at time instant t and that the control strategy $u(t)$ is a function of $x(t)$, the loss function (3.16) is then minimized by the control strategy

$$u(t) = -g(t)x(t) \qquad\qquad t = 0, 1, \ldots, N-1 \tag{3.17}$$

where $g(t)$ is equal to $k(N-t)$ given in (3.3)-(3.6) on condition that p_0 and σ^2 are replaced with q_0 and, respectively, γ^2. Therefore, also in this case, as $t \to \infty$ the gain $g(N-t) = k(t)$ converges to the golden section α.

A more general case of linear quadratic control is when the state $x(t)$ is not available for the computation of $u(t)$, and instead corrupted measurements of the state are available:

$$\begin{cases} x(t+1) &= x(t) + u(t) + w(t) \\ y(t) &= x(t) + v(t) \end{cases} \tag{3.18}$$

where the noises $w(t)$ and $v(t)$ are the same as in (3.1). This is the so called Linear Quadratic Gaussian (LQG) control problem [10]. The objective is to minimize the average quadratic loss function:

$$J_N = E\left[q_0\, x^2(N) + \sum_{t=0}^{N-1} \gamma^2(x^2(t) + u^2(t))\right] \tag{3.19}$$

It is well-known that, in this case, the loss function (3.19) is minimized by the following control strategy

$$u(t) = -g(t)\hat{x}(t) \tag{3.20}$$

where the estimate $\hat{x}(t)$ is given by the Kalman filter (3.2). The LQG control problem has the property that estimation and control are separated problems. Because of the separation principle of the LQG control, the previous connection between the Fibonacci sequence and both the Kalman filter and Linear Quadratic control carries over also to this case. In particular, both the gains k and g converge to the golden section $\alpha = (\sqrt{5} - 1)/2$. Similar results can obviously be obtained for the generic scalar system given in (3.12) by exploiting, also in this case, the duality principle.

4 Conclusions

The Fibonacci sequence and the golden ratio exhibit interesting mathematical properties which, in turn, make them appear in nature (e.g. the golden spiral built according to the golden ratio rules the growing patterns of shells, flowers [12], hurricanes and galaxies), have several uses in art and music (e.g. golden proportions in buildings [13], human figures in paintings and music compositions) as well as have many successful applications in engineering (e.g. in signal processing [7], [14]-[18], in image processing [8], in computer science [9], in optimization [11]). This paper has explored a further connection with two of the most widely used techniques in engineering, i.e. the *best linear minimum mean squared error state estimator* better known as *Kalman filter* and the *linear quadratic controller*. It has been shown that for a scalar random walk system with the two noise sources having the same power, the recursive and steady-state Kalman filters are fully governed by Fibonacci and, respectively, golden numbers. Furthermore, it has been shown that, for a generic scalar system, there exists a relationship between the system dynamics and the ratio of process-to-measurement noise variances for which the connection between the Fibonacci sequence and the Kalman filter is preserved. Finally, by exploiting the duality principle between control and estimation, a similar connection between the Fibonacci numbers and the linear quadratic controller has been outlined.

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